

Splitting fields and general differential Galois theory*

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Abstract

We present general Galois theory of nonlinear partial differential equations. For each system of differential equations we define its splitting field and differential Galois group. The main result is the theorem about Galois correspondence for normal extensions. An algebraic technique concerning constructed differential fields is presented.

1 Introduction

We present the general Galois theory of nonlinear partial differential equations. There are a lot of works devoted to different kinds of differential Galois theory. The reader should take a look at detailed review by Cassidy and Singer [5, pp. 135-137]. We present most general approach to differential Galois theory. Namely, we introduce the notion of splitting field for arbitrary systems of differential equations. For such fields and their groups of automorphisms we prove general theorem about Galois correspondence. Cogent advantage of the theory is the ability to deal with nonlinear differential equation. However, so general point of view has a disagreeable defect. The splitting fields and their differential Galois groups can be so complicated that there is no ability to provide them by useful additional structure, for example by a topology.

The work consists of three key parts. The first one contains sections 2 and 3, we go through some technical considerations that will facilitate our work in the subsequent sections. Mostly we deal with differential ideals in tensor products of differential rings and with differential closures based on the mentioned ideals. The question of existence and uniqueness of differential closure was solved by Kolchin in [3]. However, later proofs appeared in model theory [6] carry more useful information than the original ones. The second part consists of Sections 4 and 5 and is devoted to translation of the mentioned results to the algebraic language. Due to algebraic analogue of the results from model theory we define the notion of splitting field for an arbitrary system of differential equations and its differential Galois group. The last three sections form the third part devoted to scrutinizing new notions and their relations with the known ones.

* XY -pic package is used

In detail. Section 2 is devoted to terms and notation. Section 3 is a heart of our machinery. It consists of three parts where we construct special kinds of prime differential ideals. Section 3.1 is devoted to constructing a prim differential ideal with a residue field without new constant elements (theorem 1). The application to Picard-Vessiot theory is given (statement 2) also we build differential closure with algebraic field of constants (theorem 3). In section 3.2 we consider prime differential ideals with more delicate properties (theorem 8). As in the previous section we build a differential closure based on the ideals obtained (theorem 9). Section 3.3 is devoted to prime differential ideals satisfying some universal property (theorem 10). Such ideals allows to produce constructed fields and constructed differential closures (theorem 13). Section 4 is devoted to constructed fields. Our main aim is to translate to algebraic language Ressayre's theorem [6, chapter 10, sec. 4, theor. 10.18] (theorem 23). In Section 5 we translate to algebraic language theorem [6, chapter 18, sec. 1, theor. 18.1] (theorem 25). Due to results of section 4 and 5 we define in section 6 the notion of splitting field for arbitrary system of differential equations. In the first subsection of the section we introduce the notion of abstract splitting field and scrutinize its properties and relations with known ones. Statements 28 and 29 are devoted to the existence and the uniqueness. In the second subsection we study the relations between different splitting fields. This leads us to the notion of normal extension presented in the third subsection of the section. Normal extensions are in the center of our interests. In section 7 we prove main theorem about Galois correspondence (theorem 44) and give the example illustrating the behavior of the new objects. The last section gives us a geometric interpretation of locally closed points of differential spectrum of differentially finitely generated algebra over a field (theorem 46).

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2 Terms and notation

Throughout the text we assume that some differential field K of zero characteristic is fixed. Its subfield of constants is denoted by C . All differential rings are assumed to be algebras over K with a finitely many pairwise commuting derivations

$$\Delta = \{ \delta_1, \dots, \delta_m \}.$$

Differentially finitely generated algebras over K are called differentially finitely generated rings. A simple ring mean a differential ring with no nontrivial differential ideals. All undenoted tensor products are over K . A differential spectrum of a ring A is denoted by $\text{Spec}^\Delta A$. The set of all locally closed points of differential spectrum is denoted by $\text{SMax}^\Delta A$. A residue field of a prime ideal \mathfrak{p} is a field $(A/\mathfrak{p})_{\mathfrak{p}}$. Notation $\text{Qt}(B)$ denotes a

field of fraction of some domain B . All undefined terms and notation from commutative algebra are the same as in [1] and from differential algebra are as in [2].

3 Ideals in tensor products

The section is devoted to constructing different kinds of differential ideals in tensor products of differential rings. Let $\{B_\alpha\}_{\alpha \in \Lambda}$ be a family of simple differential K -algebras and suppose additionally that B_α are differentially finitely generated over K . Consider the tensor product $R = \otimes_\alpha B_\alpha$ as a direct (inductive) limit of finite tensor products over K (all details are in [1, chapter 2, ex. 23]).

3.1 Algebraic constants

We shall construct the simplest class of differential ideals in R that in some sense is related with the constants of the field K .

Theorem 1. *There exists a prime differential ideal \mathfrak{p} in $R = \otimes_\alpha B_\alpha$ such that the constants of its residue field are algebraic over C .*

Proof. Consider the set \mathbb{S} consisting of pairs $(\otimes_{\theta \in \Theta} B_\theta, \mathfrak{p}_\Theta)$ where \mathfrak{p}_Θ is an ideal in $\otimes_{\theta \in \Theta} B_\theta$ such that its residue field has algebraic over C constants only and $\Theta \subseteq \Lambda$. The set \mathbb{S} is a particularly ordered set with respect to the order

$$(\otimes_{\theta \in \Theta_1} B_\theta, \mathfrak{p}_{\Theta_1}) \leqslant (\otimes_{\theta \in \Theta_2} B_\theta, \mathfrak{p}_{\Theta_2}) \Leftrightarrow \Theta_1 \subseteq \Theta_2, \mathfrak{p}_{\Theta_2} \cap \otimes_{\theta \in \Theta_1} B_\theta = \mathfrak{p}_{\Theta_1}$$

Since B_α is simple differentially finitely generated K -algebra the set \mathbb{S} is not empty (for example [2, chapter III, sec. 10, prop. 7(d)]). It is clear that Zorn's lemma is applicable now and let $(\otimes_{\theta \in \widehat{\Theta}} B_\theta, \mathfrak{p}_{\widehat{\Theta}})$ be its maximal element. We need to show that $\otimes_{\theta \in \widehat{\Theta}} B_\theta$ coincides with R .

Suppose it is wrong. Let us define the following

$$R_{\widehat{\Theta}} = \otimes_{\theta \in \widehat{\Theta}} B_\theta / \mathfrak{p}_{\widehat{\Theta}}, \quad B_{\widehat{\Theta}} = \otimes_{\theta \in \widehat{\Theta}} B_\theta, \quad S = B_{\widehat{\Theta}} \setminus \mathfrak{p}_{\widehat{\Theta}}.$$

Consider the ring $R' = B_{\widehat{\Theta}} \otimes B_\alpha$ and let \mathfrak{p}' be a maximal ideal contracting to $\mathfrak{p}_{\widehat{\Theta}}$. We shall show that (R', \mathfrak{p}') is in \mathbb{S} and get a contradiction with maximality. Indeed, $S^{-1}R'/\mathfrak{p}'$ is a simple differentially finitely generated algebra over a field $S^{-1}R_{\widehat{\Theta}}$. But the constants of the last one are algebraic over C . Hence, the constants of the residue field of \mathfrak{p}' are so ([2, chapter III, sec. 10, prop. 7(d)]). \square

The following statement is a simple corollary of the previous fact. (Definition of universal Picard-Vessiot extension is in [4, chapter 10, sec. 1].)

Statement 2. *For any differential field K with algebraically closed subfield of constants there exists a universal Picard-Vessiot extension.*

Proof. Let $\{B_\alpha\}$ be a family of all Picard-Vessiot extensions over K up to isomorphism. Consider a differential ideal \mathfrak{p} in $R = \otimes_\alpha B_\alpha$ satisfying the condition of theorem 1. Then the ring R/\mathfrak{p} satisfies all desired properties

of the definition [4, chapter 10, sec. 1] except possibly item (2). However, since R/\mathfrak{p} does not contain new constants, then from [4, chapter 1, sec. 3, def. 1.21 and prop. 1.22] it follows that R/\mathfrak{p} is an inductive limit of simple differential rings and therefore is simple. \square

The field K will be said to be a differentially closed if any simple differentially finitely generated algebra over K coincides with K . More interesting corollary of the result obtained is the following theorem.

Theorem 3. *Let K be a differential field. Then there exists a differentially closed field L containing K such that the constant subfield of L is the algebraic closure of C .*

Proof. Consider the set $\{B_\alpha\}$ of all simple differentially finitely generated algebras over K up to isomorphism. Let \mathfrak{p} be as in theorem 1. The residue field of \mathfrak{p} will be denoted by L_1 . In the same manner we shall obtain the field L_2 from L_1 and so on. Thus, we have ascending chain of fields

$$K = L_0 \subseteq L_1 \subseteq \dots \subseteq L_n \subseteq \dots$$

The desired field L is defined by the equality $L = \cup_k L_k$. The condition that there are only algebraic over C constants in L is satisfied.

We just need to show that L is differentially closed. Let

$$B = L\{y_1, \dots, y_n\}/\mathfrak{m}$$

be a simple differentially finitely generated algebra over L .

From Ritt-Raudenbush basis theorem it follows that there exists a finite family of differential polynomials F such that $\mathfrak{m} = \{F\}$. But all coefficients of the elements of F generate a finite set and, thus, belonging to L_k for appropriate k . From the definition of L_k it follows that there is a common zero for F in L_{k+1} . Thus there is a differential homomorphism $B \rightarrow L$ over L . Since B is a simple algebra the last homomorphism is an isomorphism. \square

3.2 Local simplicity

First of all we shall introduce one useful definition. A differential ring B will be called a locally simple if there is an element $s \in B$ such that B_s is a simple ring. A prime differential ideal \mathfrak{p} will be called a locally maximal if the algebra B/\mathfrak{p} is locally simple. Locally maximal ideals are precisely locally closed points of a differential spectrum. Kolchin called locally simple algebras a constrained algebras (see. [3]).

To obtain more delicate results, we need to develop more delicate machinery. Our main target is theorem 8 but for a start we shall prove the following technical result.

Lemma 4 (“splitting lemma”). *Let D be a differential integral domain, A and B are its differential subrings such that $D = A \cdot B$ and B is differentially finitely generated. Then there are differentially finitely generated subring C in A and an element $s \in C \cdot B$ such that*

$$D_s = A \otimes_C (C \cdot B)_s.$$

Proof. The proof is based on characteristic sets. Let b_1, \dots, b_n be a differential generators of B then this set differentially generates D over A .

Hence the ring D is of the form $A\{y_1, \dots, y_n\}/\mathfrak{p}$, where \mathfrak{p} is a prime differential ideal. Let $G = \{f_1, \dots, f_m\}$ be a characteristic set of the ideal \mathfrak{p} for some ranking, and let h be the product of all initials and separants of the elements of G . Therefore we have the equality

$$D_h = A\{y_1, \dots, y_n\}_h/[f_1, \dots, f_n].$$

The coefficients of all the elements of G generate a finite subset in A . Thus we can take a differentially finitely generated algebra C containing them. Multiplying by A over C the following short exact sequence

$$0 \rightarrow [f_1, \dots, f_m] \rightarrow C\{y_1, \dots, y_n\}_h \rightarrow C\{y_1, \dots, y_n\}_h/[f_1, \dots, f_n] \rightarrow 0$$

we get the desired. \square

From “splitting lemma” we immediately get the following corollary.

Corollary 5. *Let D be a differential integral domain, A and B are its differential subrings such that $D = A \cdot B$ and B is differentially finitely generated. Then for any element $h \in D$ there exist a differentially finitely generated ring C in A and element $s \in C \cdot B$ such that $h \in C \cdot B$ and the equality holds*

$$(A \cdot B)_{sh} = A \otimes_C (C \cdot B)_{sh}.$$

It is clear that C can be replaced by any bigger ring.

Corollary 6. *Let D be a differential integral domain, A and F are its differential subrings such that $D = A \cdot F$ and F is differentially finitely generated field. Then for any element $h \in D$ there exist a differentially finitely generated ring C in A and element $s \in C \cdot F$ such that $h \in C \cdot F$ and the equality holds*

$$(A \cdot F)_{sh} = A \otimes_C (C \cdot F)_{sh}.$$

Proof. By the data it follows that there is a differentially finitely generated subring B in F such that F is a field of fraction of B . From the previous corollary it follows that

$$(A \cdot B)_{sh} = A \otimes_C (C \cdot B)_{sh}.$$

Let $S = B \setminus \{0\}$, then

$$\begin{aligned} (A \cdot F)_{sh} &= (A \cdot S^{-1}B)_{sh} = S^{-1}((A \cdot B)_{sh}) = \\ &= S^{-1}(A \otimes_C (C \cdot B)_{sh}) = A \otimes_C (C \cdot S^{-1}B)_{sh} = A \otimes_C (C \cdot F)_{sh}. \end{aligned}$$

\square

Statement 7. *Let A and B be a differential C -algebras where C is a simple differential ring. Then the canonical mapping*

$$\mathrm{Spec}^\Delta A \otimes_C B \rightarrow \mathrm{Spec}^\Delta A \times \mathrm{Spec}^\Delta B$$

is surjective.

Proof. Let S be the set of all nonzero elements of C . Since C is simple then for any differential C -algebra R there is the equality

$$\mathrm{Spec}^\Delta R = \mathrm{Spec}^\Delta S^{-1}R.$$

Hence it suffices to show the result in the case C is a field. Let $(\mathfrak{p}, \mathfrak{q})$ be a pair of differential ideals in the mentioned product. Let $S = A \setminus \mathfrak{p}$ and $T = B \setminus \mathfrak{q}$. Then from [1, chapter 3, ex. 21] and properties of tensor product it is easily follows that preimage of the pair under the mentioned mapping is naturally homeomorphic to

$$\mathrm{Spec}^\Delta S^{-1}(A/\mathfrak{p}) \otimes_C T^{-1}(B/\mathfrak{q})$$

Since C is a field the last ring is nonzero and hence its differential spectrum is not empty. \square

Theorem 8. *There exists a prime differential ideal \mathfrak{p} in $R = \otimes_\alpha B_\alpha$ such that any differentially finitely generated subalgebra B in a residue field of \mathfrak{p} is locally simple.*

Proof. Let \mathbb{S} be a set of all pairs $(\otimes_{\theta \in \Theta} B_\theta, \mathfrak{p}_\Theta)$ where \mathfrak{p}_Θ is a prime differential ideal of $\otimes_{\theta \in \Theta} B_\theta$, $\Theta \subseteq \Lambda$, and every differentially finitely generated subalgebra B in the residue field of the ideal \mathfrak{p}_Θ is locally simple. The set \mathbb{S} is a particular ordered set with respect to the order

$$(\otimes_{\theta \in \Theta_1} B_\theta, \mathfrak{p}_{\Theta_1}) \leqslant (\otimes_{\theta \in \Theta_2} B_\theta, \mathfrak{p}_{\Theta_2}) \Leftrightarrow \Theta_1 \subseteq \Theta_2, \mathfrak{p}_{\Theta_2} \cap \otimes_{\theta \in \Theta_1} B_\theta = \mathfrak{p}_{\Theta_1}$$

It should be noted that \mathbb{S} is not empty because for any α a pair $(B_\alpha, 0)$ satisfies the mentioned condition ([2, chapter III, sec. 10, prop. 7(b)]). Due to the Zorn's lemma there is a maximal element $(\otimes_{\theta \in \hat{\Theta}} B_\theta, \mathfrak{p}_{\hat{\Theta}})$ in \mathbb{S} . We need to show that $\otimes_{\theta \in \hat{\Theta}} B_\theta$ coincides with whole R . Suppose not, let us define the following

$$R_{\hat{\Theta}} = \otimes_{\theta \in \hat{\Theta}} B_\theta / \mathfrak{p}_{\hat{\Theta}}, \quad B_{\hat{\Theta}} = \otimes_{\theta \in \hat{\Theta}} B_\theta, \quad S = B_{\hat{\Theta}} \setminus \mathfrak{p}_{\hat{\Theta}}, \quad A = S^{-1} B_{\hat{\Theta}}.$$

and consider the ring $R' = B_{\hat{\Theta}} \otimes B_\alpha$. There is a maximal ideal \mathfrak{p}' in R' contracting to $\mathfrak{p}_{\hat{\Theta}}$. We shall show that (R', \mathfrak{p}') is in \mathbb{S} and this will be a contradiction.

Let the ring $S^{-1}(R'/\mathfrak{p}')$ is denoted by D and B be an arbitrary differentially finitely generated subalgebra in a residue field of \mathfrak{p}' . Then there is an element $h \in D$ such that $B \subseteq D_h$. Let $B' = B \cdot B_\alpha$. From our construction it follows that

$$D_h = (S^{-1} A \cdot B')_h.$$

Using corollary 5, we get

$$D_{sh} = S^{-1} A \otimes_C (C \cdot B')_{sh},$$

for sufficient s and C . But C is a differentially finitely generated algebra in $S^{-1} A$, therefore C is locally simple. Inverting one element more we assume that C is simple. Then statement 7 guarantees that differentially finitely generated algebra $(C \cdot B')_{sh}$ is simple. Then proposition [2, chapter III, sec. 10, prop. 7(b)] implies than B is locally simple, contradiction. \square

It should be noted that the ideal constructed in theorem 8 is a partial case of the ideals from theorem 1. In other words the quotient ring also has no non algebraic constants. There is an application of previous theorem.

Theorem 9. *Let K be a differential field. Then there exists a differentially closed field L containing K such that any differentially finitely generated subalgebra of L is locally simple.*

Proof. We should repeat the first half of the proof of theorem 3. Let $\{B_\alpha\}$ be a family of all simple differentially finitely generated algebras over K up to isomorphism and \mathfrak{p} be an ideal in $R = \otimes_\alpha B_\alpha$ as in theorem 8. The field L_1 is defined as the residue field of the ideal \mathfrak{p} . Repeating this construction for L_1 we get L_2 and etc. Therefore we have the following sequence

$$K = L_0 \subseteq L_1 \subseteq \dots \subseteq L_n \subseteq \dots$$

The desired field L is defined by the equality $L = \cup_k L_k$. The proof that L is differentially closed is similar to that of theorem 3 and therefore omitted. Let show that the obtained field satisfies the desired property.

We shall do it by the induction on k . The field L_1 satisfies required property by the definition. Suppose that hypotheses is proven for k . Let B be any differentially finitely generated subalgebra in L_{k+1} then the algebra $L_k \cdot B$ is locally simple because of the definition of L_{k+1} . Let an element h be taken such that $(L_k \cdot B)_h$ is simple. The corollary from “splitting lemma” and induction hypotheses imply that

$$(L_k \cdot B)_{sh} = L_k \otimes_C (C \cdot B)_{sh},$$

for appropriate s and C . As in the previous theorem we can assume that C is a simple differentially finitely generated algebra. Hence from statement 7 it follows that $C \cdot B$ is locally simple. Therefore proposition [2, chapter III, sec. 10, prop. 7(b)] guaranties that B is simple. \square

3.3 Universal extensions

A field constructed in theorem 9 is very close to differential closure. Let us recall the definition. Differentially closed field L containing K is called a differential closure of K if for every differentially closed field D containing K there is an embedding of L onto D . It should be noted that the mentioned embedding is not unique. Therefore, using our language, we need to prove existence and uniqueness of the differential closure. Let us note that we are interested not only in proving the known results [3] but in developing an appropriate technique that allows us to define the notion of splitting field. Constructing a differential closure is the first step on our way.

Let us recall that we are considering the ring $R = \otimes_{\alpha \in \Lambda} B_\alpha$. Assume that Λ is well-ordered. Let define a well-ordered family of rings $\{R_\alpha\}$ as follows: 1) $R_0 = B_0$, 2) $R_{\alpha+1} = R_\alpha \otimes B_{\alpha+1}$, 3) $R_\alpha = \cup_{\beta < \alpha} R_\beta$ for limit ordinals α . If \mathfrak{p} is an ideal of R then its contraction to R_α will be denoted by \mathfrak{p}_α .

Theorem 10. *Let Λ be a well-ordered set. Then there exists a prime differential ideal \mathfrak{p} such that any nonzero ideal of $R_{\alpha+1}/\mathfrak{p}_{\alpha+1}$ contracts to a nonzero ideal of $R_\alpha/\mathfrak{p}_\alpha$.*

Proof. The proof is surprisingly simple. We shall construct ideals \mathfrak{p}_α in R_α using transfinite induction. Let $\mathfrak{p}_0 = 0$. For any limit ordinal we put $\mathfrak{p}_\alpha = \bigcup_{\beta < \alpha} \mathfrak{p}_\beta$.

Let \mathfrak{p}_α is defined. Let $\mathfrak{p}_{\alpha+1}$ be a maximal ideal in $R_{\alpha+1}$ contracting to \mathfrak{p}_α . The ideal \mathfrak{p} is defined as $\bigcup_\alpha \mathfrak{p}_\alpha$. From the definition of \mathfrak{p} it follows that it satisfies all desired properties. \square

Statement 11. *Let \mathfrak{p} be an ideal as in theorem 10 then for any differentially closed field L containing K there is a differential homomorphism over K from the residue field of \mathfrak{p} to L .*

Proof. Let us make one simple remark. For any subfield F of differentially closed field L and any simple differentially finitely generated over F algebra B there is en embedding of B to L over F .

Let K_α be the residue field of the ideal \mathfrak{p}_α . It is cleat that the residue field of \mathfrak{p} is a union of the residue fields of \mathfrak{p}_α . Using transfinite induction we shall construct the desired embedding. From our remark it follows that there is an embedding of R_0 to L over K and thus an embedding of K_0 to L . Let K_α is mapped to L , then by the definition of \mathfrak{p} the algebra

$$K_\alpha \cdot (R_{\alpha+1}/\mathfrak{p}_{\alpha+1}) = S^{-1} R_{\alpha+1}/\mathfrak{p}_{\alpha+1}$$

is a simple algebra, where $S = R_\alpha \setminus \mathfrak{p}_\alpha$. Therefore, $K_{\alpha+1}$ is embedded to L . For any limit ordinal the embedding is obtained automatically. \square

Corollary 12. *The ideal \mathfrak{p} from theorem 10 satisfies the property of theorem 8.*

Proof. Let L be a differentially closed field containing K as in theorem 9. From the previous lemma it follows that the residue field of the ideal \mathfrak{p} can be embedded in L and thus satisfies the desired property. \square

The following theorem describe the structure of a differential closure.

Theorem 13. *Let K be a differential field. Then there exist a differentially closed field L and a well-ordered chain of differential fields $\{L_\alpha\}$ such that 1) $L_0 = K$, 2) $L = \bigcup_\alpha L_\alpha$, 3) $L_{\alpha+1}$ is differentially finitely generated over L_α , 4) every differentially finitely generated over L_α sub-algebra in $L_{\alpha+1}$ is locally simple. The mentioned field L is a differential closure of K .*

Proof. Let $\{B_\alpha\}$ be a family of all simple differentially finitely generated algebras over K up to isomorphism. Let a well-ordering on Λ be fixed and ideal \mathfrak{p} be as in theorem 10. The field K_1 is defined as the residue field of the ideal \mathfrak{p} . Let $K_{1\alpha}$ be the residue field of the ideal \mathfrak{p}_α . Repeating this construction for K_1 we obtain K_2 and so on. Thus we have the following ascending chain of the fields

$$K = K_0 \subseteq K_1 \subseteq \dots \subseteq K_n \subseteq \dots$$

Let $L = \cup_k K_k$. Merging all families $K_{k\alpha}$ in one ascending chain of the fields from [2, chapter III, sec. 10, prop. 7(b)] it follows that we get the desired chain of fields on L .

The fact that L is differentially closed has the same proof as in theorem 3. Since by the definition the universal property for K_{k+1} over K_k holds (statement 11), then this property holds for whole field L . \square

4 Constructed fields

The section is devoted to scrutinizing the fields constructed in theorem 13. Let a differential field L and a sequence of its differential subfields $\{L_\alpha\}$ be given such that the following holds: 1) $L_0 = K$, 2) $L = \cup_\alpha L_\alpha$, 3) $L_{\alpha+1}$ differentially finitely generated over L_α , 4) any differentially finitely generated over L_α algebra in $L_{\alpha+1}$ is locally simple. In that case we shall say that L is constructed over K and that $\{L_\alpha\}$ is a construction for L over K . Terms are taken from [6, chapter 10, sec. 4]. We shall start from simple statement.

Statement 14. *Let a differential field L and a sequence of its differential subfields $\{L_\alpha\}$ be given such that the following holds: 1) $L_0 = K$, 2) $L = \cup_\alpha L_\alpha$, 3) $L_{\alpha+1}$ differentially not more than countably generated over L_α , 4) any differentially finitely generated over L_α algebra in $L_{\alpha+1}$ is locally simple. Moreover, the sequence $\{L_\alpha\}$ can be extended to a construction of L .*

Proof. It suffices to extend one stage $L_\alpha \subseteq L_{\alpha+1}$. Let $\{x_n\}_{n=0}^\infty$ be a system of differential generators of $L_{\alpha+1}$ over L_α . Let us define the sequence of L_α -algebras B_n and its fields of fractions F_n as follows:

$$B_n = L_\alpha\{x_1, \dots, x_n\}.$$

Let us show that the property (4) holds for F_{n+1} over F_n . Then we shall obtain the desired construction.

Let B be differentially finitely generated F_n -algebra in F_{n+1} . Then it is of the form

$$B = F_n\{z_1, \dots, z_k\}$$

for appropriate $z_1, \dots, z_k \in F_{n+1}$. Hence, B is a localization of

$$B_n\{z_1, \dots, z_k\}.$$

By the data the last algebra is locally simple. Since localization of locally simple algebra is locally simple, we get the desired result. \square

The following step is the condition that guarantees that the field is constructed over another one. We shall introduce some auxiliary terms borrowed from [6, chapter 10, sec. 4].

Let L be a constructed field. Then from conditions (3) and (4) for any ordinal α there is a simple differentially finitely generated ring $B_\alpha \subseteq L_{\alpha+1}$ such that $L_{\alpha+1}$ is a field of fraction of $L_\alpha \cdot B_\alpha$. The family $\{B_\alpha\}$ we shall call a family of generators for constructed field L . It should be noted that

this family is not unique. The field L_α can be restored by the family of generators as follows

$$L_\alpha = K\langle \{B_\beta\}_{\beta < \alpha} \rangle$$

or

$$L_{\alpha+1} = L_\alpha \langle B_\alpha \rangle.$$

Statement 15. Let L be a constructed field, $\{B_\alpha\}$ be a family of generators. Then for any B_α there are a finite family $\{B_{\alpha_1}, \dots, B_{\alpha_t}\}$ with condition $\alpha_i < \alpha$ and the element $s \in F_\alpha \cdot B_\alpha$ such that

$$(L_\alpha \cdot B_\alpha)_s = L_\alpha \otimes_{F_\alpha} (F_\alpha \cdot B_\alpha)_s,$$

where

$$F_\alpha = K\langle B_{\alpha_1}, \dots, B_{\alpha_t} \rangle$$

and algebra $(F_\alpha \cdot B_\alpha)_s$ is simple.

Proof. Algebra $L_\alpha \cdot B_\alpha$ is locally simple, therefore there is an element $h \in L_\alpha \cdot B_\alpha$ such that the algebra

$$(L_\alpha \cdot B_\alpha)_h$$

is simple. Then from the corollary from “splitting lemma” it follows that there exist a differentially finitely generated ring $C \subseteq L_\alpha \cdot B_\alpha$ and an element $s' \in C \cdot B_\alpha$ such that

$$(L_\alpha \cdot B_\alpha)_{s'h} = L_\alpha \otimes_C (C \cdot B_\alpha)_{s'h}.$$

Let us denote $s'h$ by s . Since the ring C differentially finitely generated than there is a finite family of rings

$$\{B_{\alpha_1}, \dots, B_{\alpha_t}\}$$

such that C belongs to the field

$$F_\alpha = K\langle B_{\alpha_1}, \dots, B_{\alpha_t} \rangle,$$

the following holds

$$(L_\alpha \cdot B_\alpha)_s = L_\alpha \otimes_{F_\alpha} (F_\alpha \cdot B_\alpha)_s.$$

Since the right part of the equality is a simple differential ring, then from statement 7 it follows that $(F_\alpha \cdot B_\alpha)_s$ is simple. \square

Suppose that some generator family $\{B_\alpha\}$ of a field L be given. Let us define the notion of package for the ring B_α by the induction. The package π_0 of B_0 is an empty set. Let the notion of package be defined for all $\beta < \alpha$, we shall define the package of B_α . Let $\{B_{\alpha_1}, \dots, B_{\alpha_t}\}$ be the family as in the previous statement, then the package π_α is the following union

$$\{B_{\alpha_1}, \dots, B_{\alpha_t}\} \bigcup \cup_i \pi_{\alpha_i}.$$

The package field of B_α is the following field

$$F_\alpha = K\langle \pi_\alpha \rangle.$$

It should be noted that the notion of package is depended on the notion of generators. From the definition of package the following statement follows.

Lemma 16 (“package splitting”). *Let L be constructed, $\{B_\alpha\}$ be family of generators, and for each α some package π_α is fixed. Then for any α there is an element $s \in F_\alpha \cdot B_\alpha$ such that*

$$(L_\alpha \cdot B_\alpha)_s = L_\alpha \otimes_{F_\alpha} (F_\alpha \cdot B_\alpha)_s,$$

and the algebra $(F_\alpha \cdot B_\alpha)_s$ is simple.

Statement 17. *Let L be constructed and $\{B_\alpha\}$ be family of generators. Let for each α some package π_α be fixed, for some α a subfield F contains the field F_α , and an element s is as in “package splitting” lemma. Then algebra $(F \cdot B_\alpha)_s$ is simple.*

Proof. From “package splitting” lemma it follows that for the element s we have

$$(L_\alpha \cdot B_\alpha)_s = L_\alpha \otimes_{F_\alpha} (F_\alpha \cdot B_\alpha)_s.$$

Since F contains F_α then

$$(L_\alpha \cdot B_\alpha)_s = L_\alpha \otimes_F F \otimes_{F_\alpha} (F_\alpha \cdot B_\alpha)_s = L_\alpha \otimes_F (F \cdot B_\alpha)_s.$$

From the choice of element s it follows that the algebra at the left part of the last equality is simple. Therefore, from statement 7 it follows that the algebra $(F \cdot B_\alpha)_s$ is so. \square

Let suppose that for any ring B_α from the family $\{B_\alpha\}_{\alpha \in \Lambda}$ some package π_α is fixed. Consider subfamily of rings $\{B_\beta\}_{\beta \in X}$ ($X \subseteq \Lambda$) such that with any ring B_β its package belongs to the family. We shall call this family closed (term is taken from [6, chapter 10, sec. 4]). Let subfield $F \subseteq L$ be generated by some closed family of rings $\{B_\beta\}$, then we shall say such field is closed. The set X is a well-ordered set with respect to induced order from Λ . Then we are able to define the family of fields $\{L'_\beta\}$ as follows

$$L'_\beta = K < \{B_\theta\}_{\theta < \beta} >_{\theta \in X}.$$

The closed fields are very important because of the following two statements (the original statements are [6, chapter 10, sec. 4, st. 10.15] and [6, chapter 10, sec. 4, st. 10.17], respectively).

Statement 18. *Let L be constructed and $\{B_\alpha\}$ be family of generators. Let for each α some package π_α be fixed and and some subfield F of L is closed. Then the field F is constructed over K and its construction is $\{L'_\beta\}$.*

Proof. To prove the statement we just need to show that item (4) of the definition holds for $L'_\beta \subseteq L'_{\beta+1}$. From the definition it follows that $L'_{\beta+1} = L'_\beta < B_\gamma >$ for some $\gamma \in X$, for any $\theta \in X$ strictly less than γ we have that $B_\theta \subseteq L'_\beta$. Statement [2, chapter III, sec. 10, prop. 7(b)] says that it suffices to show that the algebra $L'_\beta \cdot B_\gamma$ is locally simple. The field F is closed, therefore it contains F_γ . From the definition of package it follows that F_γ belongs to L'_β , because it generates by the rings B_θ , where $\theta \in X$ and strictly less than γ . Then from statement 17 the desired follows. \square

Statement 19. Let L be constructed and $\{B_\alpha\}$ be family of generators. Let for each α some package π_α be fixed and some subfield F of L is closed. Then the field L is constructed over F and the construction of L is $\{F<L_\alpha>\}$.

Proof. To prove the statement we just need to show that item (4) of the definition holds for the pair $F<L_\alpha> \subseteq F<L_{\alpha+1}>$. Statement [2, chapter III, sec. 10, prop. 7(b)] says that it suffices to show that algebra

$$F<L_\alpha> \cdot B_\alpha$$

is locally simple. From the definition we know that algebra $L_\alpha \cdot B_\alpha$ is locally simple. Let s be such element that algebra

$$(L_\alpha \cdot B_\alpha)_s$$

is simple. We shall prove that algebra

$$(F<L_\alpha> \cdot B_\alpha)_s$$

is also simple. Since from previous statement F is constructed, then using induction by β we shall show that algebra

$$(L'_\beta <L_\alpha> \cdot B_\alpha)_s$$

is simple. Since the direct limit of simple algebras is simple, we only need to prove the statement in the case of non limit ordinals.

First of all we note that for any β we have the equality

$$L'_\beta <L_\alpha> = \text{Qt}(L'_\beta \cdot B_\alpha).$$

Let us denote the last field by \tilde{F} . Since the field $\text{Qt}(\tilde{F} \cdot B_\alpha)$ contains a package of $B_{\beta+1}$ then from statement 17 it follows that for some $h \in \tilde{F} \cdot B_{\beta+1}$ the algebra

$$(\text{Qt}(\tilde{F} \cdot B_\alpha) \cdot B_{\beta+1})_h$$

is simple. Let $S = (\tilde{F} \cdot B_\alpha)_s \setminus \{0\}$, then

$$\begin{aligned} (\text{Qt}(\tilde{F} \cdot B_\alpha) \cdot B_{\beta+1})_h &= (S^{-1}((\tilde{F} \cdot B_\alpha)_s) \cdot B_{\beta+1})_h = \\ &= S^{-1}(((\tilde{F} \cdot B_\alpha)_s \cdot B_{\beta+1})_h) = S^{-1}(((\tilde{F} \cdot B_{\beta+1})_h \cdot B_\alpha)_s) \end{aligned}$$

Thus the algebra

$$S^{-1}(((\tilde{F} \cdot B_{\beta+1})_h \cdot B_\alpha)_s)$$

is simple. Since $(\tilde{F} \cdot B_\alpha)_s$ is simple by induction, then the set S does not meet any differential ideal in any differential overring. Hence, the ring

$$((\tilde{F} \cdot B_{\beta+1})_h \cdot B_\alpha)_s$$

is also simple, and thus its localization

$$(L'_{\beta+1} <L_\alpha> \cdot B_\alpha)_s$$

is simple too. □

Statement 20. *Let L be constructed, then any differentially finitely generated subring in L is locally simple.*

Proof. For any constructed field L the following property holds: any differential homomorphism of the field K to a differentially closed field can be extended to L . Thus our field can be embedded to the field from theorem 9. From that the desired follows. \square

Statement 21. *Let L be constructed and $\{B_\alpha\}$ be family of generators. Let for each α some package π_α be fixed and F is a differentially finitely generated subfield in L . Then there exists a differentially finitely generated closed subfield F' containing F .*

Proof. Since F differentially finitely generated, then it belongs to the field generated by some algebras $B_{\alpha_1}, \dots, B_{\alpha_k}$. Let F' be the field generated by this family and their packages. It is clear that F' satisfies all desired properties. \square

Statement 22. *Let L be constructed, then L is constructed over any differentially finitely generated subfield.*

Proof. Let F be differentially finitely generated subfield in L . Then from the previous statement it follows that there is differentially finitely generated closed subfield F' containing F . Thus, statement 19 says that L is constructed over F' . From statement [2, chapter III, sec. 10, prop. 7(b)] it follows that F' is constructed over F . Therefore, L constructed over F . \square

The following result in model theory is known as Ressayre's theorem [6, chapter 10, sec. 4, theor. 10.18], we shall prove its algebraic analogue.

Theorem 23. *Let L and F be constructed differentially closed fields, then they are isomorphic to each other.*

Proof. Let $\{L_\alpha\}$ and $\{F_\beta\}$ be a constructions on L and F , respectively. Also we assume that a family of generators and a their packages are fixed for both fields. Then consider the following set

$$\Sigma = \{(L', F', f') \mid L' \subseteq L, F' \subseteq F, f: L' \rightarrow F'\},$$

where L' and F' are closed and f' is an isomorphism between them. This set is a particulary ordered set with respect to the following order

$$(L', F', f') \leq (L'', F'', f'') \Leftrightarrow L' \subseteq L'', F' \subseteq F'', f''|_{L'} = f'$$

The set Σ is not empty because the element (K, K, Id) belongs to it. It is clear that we can apply the Zorn lemma. Therefore there is a maximal element $(\widehat{L}, \widehat{F}, \widehat{f})$. Suppose that contrary holds, for example we have $\widehat{L} \neq L$.

Statement 19 guarantees that the field L is constructed over \widehat{L} (F over \widehat{F} , respectively). Moreover the family of generators for L over \widehat{L} coincides with that for L over K . Therefore the notion of package does not change. By the hypothesis there is a simple ring $B_\alpha \subseteq L$ not belonging

to \widehat{L} . From the definition of constructed field there exists an element $s \in \widehat{L} \cdot B_\alpha$ such that $(\widehat{L} \cdot B_\alpha)_s$ is simple. Since F differentially closed the homomorphism \widehat{f} can be extended to the field of fraction of $\widehat{L} \cdot B_\alpha$. Let us denote this field by \widehat{L}_1 . From statement 21 it follows that there is a differentially finitely generated closed field \widehat{F}_1 containing the image of \widehat{L}_1 . Statements 20 and [2, chapter III, sec. 10, prop. 7(b)] imply that the field \widehat{F}_1 is a field of fraction of some differentially finitely generated simple algebra over image of \widehat{L}_1 . Since L is differentially closed there is an embedding of \widehat{F}_1 to L . Again, statement 21 implies that there exists differentially finitely generated closed field \widehat{L}_2 containing the image of \widehat{F}_1 . Then we shall apply the same method for the field \widehat{L}_2 and so on. This construction can be expressed on the following diagram

$$\begin{array}{ccccccc} \widehat{L} & & \widehat{L}_1 & & \widehat{L}_2 & & \cdots \\ \downarrow \widehat{f} & & \downarrow & \nearrow & \downarrow & \nearrow & \cdots \\ \widehat{F} & & \widehat{F}_1 & & \widehat{F}_2 & & \cdots \\ & & & & \cdots & & \cdots \\ & & & & \widehat{F}_n & & \cdots \end{array}$$

As a result we have two sequences of fields \widehat{L}_k in L and \widehat{F}_k in F and every such field is closed. Then defining the fields as $L' = \cup_k \widehat{L}_k$ and $F' = \cup_k \widehat{F}_k$, we see that they are closed and f can be extended to them. Thus we have a contradiction with maximality. \square

5 Uniqueness theorem

Let us recall that all rings are assumed to be algebras over a field K . The following theorem contains the technique that similar to the technique in [6, chapter 18, sec. 1, prop. 18.1]. This machinery will play a major role in the following sections. First of all we need some variation of the “splitting lemma”. We are sure that the statement below is very similar to [6, chapter 16, sec. 2, coroll. 16.7, theor. 16.8].

Lemma 24. *Let A and B be arbitrary differential rings without nilpotent elements and let there is an element $h \in A \otimes B$ such that $(A \otimes B)_h$ is simple. Then B is locally closed.*

Proof. The element h is of the following form $\sum a_i \otimes b_i$. We can suppose that elements a_i are linearly independent over K . Since $A \otimes B$ is locally simple, then for any nonzero differential ideal \mathfrak{a} we have

$$((A \otimes B)/\mathfrak{a})_h = 0$$

Particularly, for any differential prime ideal \mathfrak{p} of B we have

$$(A \otimes (B/\mathfrak{p}))_h = 0$$

Since A and B/\mathfrak{p} have no nilpotent elements and the field K is of characteristic zero, then $A \otimes (B/\mathfrak{p})$ has no nilpotent elements ([4, lemma A.16, p. 363]). The condition that last ring is zero means that $h^n = 0$ in $A \otimes (B/\mathfrak{p})$ and thus $h = 0$. In other words the elements b_i belong to \mathfrak{p} . Therefore differential spectrum of B_{b_i} consists of one single element for each i , q. e. d. \square

Let L be a differential closure of K and let field D be its a constructed differential closure (the existence follows from theorem 13). It is clear that L can be embedded in D . The following problem is to find a construction on differential closure embedded in constructible field.

Theorem 25. *Let L be a differential closure of K , then L is constructed.*

Proof. From the arguments above it follows that we can suppose that L is embedded in some constructed differential closure D with construction D_α . Let us show a naive version of the “proof”. We can choose the family $L_\alpha = L \cap D_\alpha$ as a hypothetical construction. There is only one problem this family can not be a construction. To get a correct family we need to change the initial construction on D . Namely, we shall construct the family of subfields D'_α with the following conditions

1. $D_\alpha \subseteq D'_\alpha$
2. D'_α is closed
3. $D'_{\alpha+1}$ is not more than countably differentially generated over D'_α
4. $L \cdot D'_\alpha = L \otimes_{L_\alpha} D'_\alpha$, where $L_\alpha = L \cap D'_\alpha$
5. $L_{\alpha+1}$ is not more than countably differentially generated over L_α

The mentioned conditions hold under taking direct limits, thus we need to prove the result for non limit ordinals. We assume the some generator family $\{B_\alpha\}$ of D is fixed and the packages are defined. Therefore the notion of closeness is well-defined.

We shall construct a sequence of fields D_k as follows. Let $\overline{D}_1 = D'_\alpha < D_\alpha >$. It is differentially finitely generated over D'_α . By the construction \overline{D}_1 is closed, but there is no guaranty that \overline{D}_1 and L are linearly disjoint, in other words we have

$$L \cdot \overline{D}_1 \neq L \otimes_{L \cap \overline{D}_1} \overline{D}_1.$$

From corollary 6 of “splitting lemma” (using $L \cap D'_\alpha$ instead of K) it follows that there exist a differentially finitely generated over $L \cap D'_\alpha$ field C_1 and an element $s \in C_1 \cdot \overline{D}_1$ such that

$$(L \cdot \overline{D}_1)_s = L \otimes_{C_1} (C_1 \cdot \overline{D}_1)_s.$$

Putting $\overline{D}_2 = \text{Qt}(C_1 \cdot \overline{D}_1)$, we have

$$L \cdot \overline{D}_2 = L \otimes_{C_1} \overline{D}_2.$$

From the definition of tensor product we have $C_1 = L \cap \overline{D}_2$ and C_1 is differentially finitely generated over $L \cap D'_\alpha$. But now the field \overline{D}_2 is not necessarily closed. Statement 21 guarantees the existence of closed differentially finitely generated over D'_α field \overline{D}_3 containing \overline{D}_2 . Let repeat with \overline{D}_3 the actions done with \overline{D}_1 . We get the field \overline{D}_4 containing \overline{D}_3 such that

$$L \cdot \overline{D}_4 = L \otimes_{C_2} \overline{D}_4,$$

and $C_2 = L \cap \overline{D}_4$ is differentially finitely generated over $L \cap D'_\alpha$. Therefore $C_1 \subseteq C_2$. If we continue in the same manner we shall obtain the sequences of the fields \overline{D}_k and C_k with the following properties

1. C_k is differentially finitely generated over $L \cap D'_\alpha$
2. \overline{D}_k is differentially finitely generated over D'_α
3. $C_n = L \cap \overline{D}_{2n}$
4. $L \cdot \overline{D}_{2n} = L \otimes_{C_n} \overline{D}_{2n}$
5. \overline{D}_{2n+1} is closed

Let $D'_{\alpha+1} = \cup_k \overline{D}_k$. Then from the item (4) it is closed. Taking direct limit (the details in [1, chapter 2, ex. 20]), we get

$$L \cdot D'_{\alpha+1} = L \otimes_{L_{\alpha+1}} D'_{\alpha+1},$$

where $L_{\alpha+1} = L \cap D'_{\alpha+1}$ and from item (2) $L_{\alpha+1} = \cup_k C_k$. As we can see all five items declared above are satisfied.

Let us show that the sequence of fields $\{L_\alpha\}$ satisfies the properties of statement 14. For that we only need to check the condition (4). Let B be a differentially finitely generated algebra over L_α , then algebra $D'_\alpha \cdot B$ is differentially finitely generated over D'_α . Since D'_α is closed statements 19 and 20 imply that $D'_\alpha \cdot B$ is locally simple. Now consider the following sequence of inclusions

$$D'_\alpha \otimes_{L_\alpha} B \subseteq D'_\alpha \otimes_{L_\alpha} L = D'_\alpha \cdot L$$

Therefore

$$D'_\alpha \cdot B = D'_\alpha \otimes_{L_\alpha} B.$$

Since $D'_\alpha \otimes_{L_\alpha} B$ is locally simple, then from previous lemma (with L_α instead of K) it follows that B is also locally simple. \square

Corollary 26. *Differential closure is unique up to isomorphism and constructed.*

6 Splitting fields

6.1 Abstract splitting fields

As before all rings are assumed to be algebras over a field K . Consider a family of differentially finitely generated rings $\{B_\alpha\}$ (not necessarily simple) and let L be a field. Consider all homomorphisms $B_\alpha \rightarrow L$ for all α . Thereby some family of subrings in L is defined. If L is the smallest field containing this family, then we shall say that L is generated by $\{B_\alpha\}$. The image of B_α in L under some homomorphism we shall denote by \overline{B}_α .

We shall say that L is a splitting field of the family $\{B_\alpha\}$ over K if the following conditions are satisfied:

1. (L is big enough) For any α and any locally maximal ideal \mathfrak{m} in $L \otimes B_\alpha$ the condition $(L \otimes B_\alpha)/\mathfrak{m} = L$ holds ($l \otimes 1 \mapsto l$).
2. (L is not so big) The field L is generated by the family $\{B_\alpha\}$.
3. (L is universal) For any field L' satisfying conditions (1) and (2) there is an embedding of L to L' over K .

Note that the differential closure is a splitting field for the family of all differentially finitely generated algebras over K .

Statement 27. Let $\{B_\alpha\}$ be a family of differentially finitely generated rings and let a field L satisfying the following conditions

1. For any α and for any locally maximal ideal \mathfrak{m} in $L \otimes B_\alpha$ the following holds $(L \otimes B_\alpha)/\mathfrak{m} = L$.
2. The field L is generated by $\{B_\alpha\}$.
3. L is constructed over K .

Then L is splitting field of the family $\{B_\alpha\}$ over K .

Proof. We shall show that if L is constructed then it satisfies the condition (3) of the definition of splitting field. Indeed, Let L' be arbitrary field satisfying the conditions (1) and (2) then let \overline{L}' be its differential closure. Since L is constructed there is an embedding of L to \overline{L}' . We need to show that the image of L is in L' . Since L is generated by \overline{B}_α then it suffices to show that every such a ring is in L' . The algebra $L' \cdot \overline{B}_\alpha$ is locally simple because of statement 20, but now from the definition of L' our algebra coincides with L' , q. e. d. \square

Let us show that for arbitrary family of rings there exists a unique splitting field and that the splitting field is of the special form.

Statement 28. For any family of rings $\{B_\alpha\}$ there exists a constructed splitting field. Moreover a differential closure of splitting field coincides with a differential closure of K .

Proof. Consider a family of simple differential rings of the following form

$$F_1 = \{(B_\alpha/\mathfrak{m})_s\}.$$

Let this family be well-ordered. Then choose in

$$R_1 = \otimes_{B \in F_1} B$$

an ideal \mathfrak{p} as in theorem 10. The field of fraction of an ideal \mathfrak{p}_1 will be denoted by L_1 . Now we need to repeat the construction for the family $\{L_1 \otimes B_\alpha\}$ instead of $\{B_\alpha\}$. We shall get a field L_2 containing L_1 . Let $L = \cup_k L_k$. We shall prove that L is a desired one. Note that L is constructed.

We need to check the conditions (1) and (3) of the definition. Let us show the first one. Let

$$(L \cdot \overline{B}_\alpha)_s = (L \otimes B_\alpha)_s / \mathfrak{m}$$

be a simple algebra. Then from corollary 5 it follows that there exist differentially finitely generated ring $C \subseteq L$ and an element $h \in C \cdot \overline{B}_\alpha$ such that

$$(L \cdot \overline{B}_\alpha)_{sh} = L \otimes_C (C \cdot \overline{B}_\alpha)_{sh}.$$

Since C differentially finitely generated then for some k we have $C \subseteq L_k$. Therefore

$$(L \cdot \overline{B}_\alpha)_{sh} = L \otimes_{L_k} (L_k \cdot \overline{B}_\alpha)_{sh}.$$

But from the definition it follows that the algebra $(L_k \cdot \overline{B}_\alpha)_{sh}$ can be embedded to L_{k+1} , and thus to L . In other words the algebra $(L \cdot \overline{B}_\alpha)_{sh}$ can be embedded to L and therefore is isomorphic to L , q. e. d.

It is clear that since L is constructed over K then differential closure of L is a differential closure of K . Now from statement 27 it follows that L is a desired field. \square

Statement 29. *For any family of rings $\{B_\alpha\}$ the splitting field is unique up to isomorphism. Moreover, every automorphism of K can be extended to an automorphism of the splitting field.*

Proof. Let L and L' be two splitting fields. Then from previous statement and corollary 26 it follows that we can identify the following fields

$$\overline{L} = \overline{L}' = \overline{K},$$

namely, it can be done using the following mappings

$$\begin{array}{ccccc} \overline{L} & \xrightarrow{\quad} & \overline{K} & \xleftarrow{\quad} & \overline{L}' \\ \cup & & \cup & & \cup \\ L & & K & & L' \\ \cup & & \cup & & \cup \\ K & = & K & = & K \end{array}$$

One thing we need to check is that the fields L and L' coincide. Let show the inclusion $L \subseteq L'$. Since the family \overline{B}_α generates L over K then it suffices to show that any algebra B of the mentioned form belongs to L' . Indeed, consider the algebra $L' \cdot B$, then from the equality $\overline{L} = \overline{K}$ and statement 20 it follows that our algebra is locally simple and therefore B belongs to L' from the definition. Another inclusion is checked in analogue way.

It should be noted that any automorphism of K can be extended to an automorphism of \overline{K} , it is another point of view on the uniqueness theorem. But now, identifying splitting fields by the constructed isomorphism we reduce the problem to the previous one. \square

From the definition of splitting field of $\{B_\alpha\}$ it follows that for any locally maximal ideal \mathfrak{m} in B_α the ring B_α/\mathfrak{m} can be embedded to the splitting field. Let us show that more general fact holds.

Statement 30. *If L is a splitting field of $\{B_\alpha\}$, then for any locally maximal ideal \mathfrak{m} in*

$$L \otimes B_{\alpha_1} \otimes \cdots \otimes B_{\alpha_n}$$

its residue field coincides with L . As a corollary, any locally simple algebra of the form

$$(B_{\alpha_1} \otimes \cdots \otimes B_{\alpha_n})/\mathfrak{m}$$

can be embedded to L .

Proof. We shall prove using induction by the number of multiples. The case $n = 1$ follows from the definition of splitting field. Let suppose that we proved for the case n . Consider the equality

$$(L \otimes B_{\alpha_1} \otimes \cdots \otimes B_{\alpha_{n+1}})/\mathfrak{m} = ((L \otimes B_{\alpha_1} \otimes \cdots \otimes B_{\alpha_n})/\mathfrak{m}^c \otimes B_{\alpha_{n+1}})/\mathfrak{m}$$

From statement [2, chapter III, sec. 10, prop. 7(b)] it follows that \mathfrak{m}^c is locally maximal ideal, therefore by the induction the last ring is equal to

$$(L \otimes B_{\alpha_{n+1}})/\mathfrak{m} = L,$$

q. e. d.

Denote the algebra of the mentioned form by B , then for some s from B the algebra B_s is simple. Therefore for any locally maximal ideal \mathfrak{m} we have the equality

$$(L \otimes B_s)/\mathfrak{m} = L.$$

The last equation guarantees an embedding of B to L . \square

Statement 31. *Let F be a subfield of a splitting field L of the family $\{B_\alpha\}$ such that L is constructed over F . Then L is a splitting field over F of a family $\{F \otimes B_\alpha\}$.*

Proof. The first item of the definition follows from the equality

$$L \otimes_F F \otimes B_\alpha = L \otimes B_\alpha.$$

Since L is generated by B_α , then it is generated by $F \otimes B_\alpha$. Now the condition (3) follows from statement 27. \square

A group of differential automorphisms of L over K we shall denote by $\text{Aut}^\Delta(L/K)$. For arbitrary subgroup H in $\text{Aut}^\Delta(L/K)$ by L^H we shall denote the subfield of invariant under H elements.

Statement 32. *For any splitting field L the following equality holds*

$$K = L^{\text{Aut}^\Delta(L/K)}.$$

Proof. Let $x \in L \setminus K$, then it belongs to some algebra of the form

$$B = \overline{B}_{\alpha_1} \cdot \dots \cdot \overline{B}_{\alpha_n}.$$

Since the algebra is differentially finitely generated and L is constructed, then statement 20 guarantees that B is locally simple. In other words, there is an element $s \in B$ such that $D = B_s$ is simple. Since D is a Ritt algebra, then the ring $D \otimes D$ has no nilpotent elements [4, lemma A.16, p. 362]. Let \mathfrak{m} be a maximal differential ideal in

$$(D \otimes D)_{1 \otimes x - x \otimes 1}.$$

Let the residue field of \mathfrak{m} will be denoted by F and let D_1 be the image of the first multiple in F and D_2 of the second one. The identity homomorphism $\text{Qt}(D_1) \rightarrow \text{Qt}(D) \subseteq L$ such that $x \otimes 1 \mapsto x$ can be extended to an automorphism of F (statement 29). So we can assume that D_1 and D_2 are in L and that $D_1 = D$. The image of element $1 \otimes x$ in L we shall denote by x' . From statement 29 it follows that the isomorphism of $\text{Qt}(D_1)$ to $\text{Qt}(D_2)$ such that $x \mapsto x'$ can be extended to an isomorphism of L . \square

Let us define the following transformation of the family $\{B_\alpha\}$. For any ring B_α consider the family $X_\alpha = \{B_\alpha/\mathfrak{m}\}$ whenever \mathfrak{m} is a locally maximal ideal of B_α . Consider the family $X = \cup_\alpha X_\alpha$. This family consists of locally simple rings only. Now, for each algebra $B \in X$ there is an element $s \in B$ such that B_s is simple. Let Y be a family of all B_s of the mentioned form.

Statement 33. *Let $\{B_\alpha\}$ be arbitrary family of rings and the families X and Y are obtained by the described method. Then the splitting fields of these three families naturally coincide.*

Proof. To prove it suffices to see that splitting fields of the families $\{B_\alpha\}$ and $\{\overline{B}_\alpha\}$ coincide. But the algebras \overline{B}_α are locally simple (it follows from statement 20 and the fact that splitting field is constructed). \square

Consider the ring of differential polynomials $K\{y_1, \dots, y_n\}$ and let f_1, \dots, f_k be a family of differential polynomials. Then a splitting field of the system

$$\begin{cases} f_1(y_1, \dots, y_n) = 0 \\ \dots \\ f_k(y_1, \dots, y_n) = 0 \end{cases}$$

is called a splitting field of the following algebra

$$B = K\{y_1, \dots, y_n\}/[f_1, \dots, f_k].$$

It should be noted that the notion of splitting field is in agreement with the existing ones. Namely, it is clear that:

1. If f is a polynomial of one variable (the element of $K[x]$), then its splitting field in usual sense coincides with our splitting field. Derivation are assumed to be zero.
2. The splitting field of a linear differential equation coincides with a Picard-Vessiot extension [4].
3. The splitting of linear differential equation with parameters coincides with parameterized Picard-Vessiot extension [5].
4. Strongly normal extensions in the sense of Kolchin are the splitting fields of the corresponding equations [8].

6.2 Splitting subfields

Let \overline{K} be a differential closure of K and L be a splitting field. Then from condition (3) of the definition it follows that there is an embedding of L to \overline{K} . However, there are a lot of such embeddings. In a view of the aforesaid we shall emphasize the following statement.

Statement 34. *Let L be a splitting field of a family $\{B_\alpha\}$. Then there is a largest subfield in \overline{K} isomorphic to L . This field coincides with a minimal field generated by algebras of the form \overline{B}_α .*

Proof. Let L' be a subfield in \overline{K} generated by all algebras of the form \overline{B}_α . We need to show that L' is a splitting field. As we know (statement 28) the field \overline{L} is isomorphic to \overline{K} . Let us show that L goes to L' under this isomorphism. Since L is generated by algebras of the form \overline{B}_α , then the image of L is in L' . Conversely, For any algebra B of the form \overline{B}_α it follows that $L \cdot B$ is locally simple. Therefore from the definition it follows that B is in L . \square

Subfield from previous statement we shall call a splitting subfield. Let us introduce the following class of the subfields: a subfield L of \overline{K} will be called “good” if \overline{K} is a differential closure of L , or in light of corollary 26 the field \overline{K} is constructed over L . It should be noted that not all subfields are “good”. However, the following result holds.

Statement 35. *Every splitting subfield L is “good”.*

Proof. Since differential closure of L coincides with differential closure of K (statement 28) we get the desired. \square

Full differential Galois group $\text{Gal}^\Delta(\overline{K}/K)$ of the field K is a group of all differential automorphisms of \overline{K} over K . Since the field K is fixed we shall denote this group by G . We shall say that subfield L of \overline{K} is invariant if $G(L) \subseteq L$.

Statement 36. *Subfield L is invariant iff L is a splitting subfield.*

Proof. It is clear that subfield is invariant iff with any simple differential subalgebra B it contains all algebras that isomorphic to B . But from statement 34 it follows that such fields are exactly splitting fields. \square

Corollary 37. *If L is invariant then \overline{K} is constructed over L*

Proof. If the field L is invariant then from previous statement it follows that L is splitting subfield. Then statement 35 finishes the proof. \square

6.3 Normal extensions

The section is devoted to differential fields belonging not to a differential closure of a fixed field but to arbitrary splitting field. We shall say that field extension $F \subseteq L$ is normal if L is a splitting field of some family over F .

Statement 38. *A field extension $F \subseteq L$ is normal iff L can be embedded to \overline{F} as an invariant subfield.*

Proof. Let $F \subseteq L$ is normal, then from statement 28 it follows that differential closure of L coincides with \overline{F} . Let us show that L is invariant. Let L be a splitting field of the family $\{B_\alpha\}$ and let B is a subalgebra in \overline{F} which is isomorphic to some \overline{B}_α . Then from the statement 20 it follows that the algebra $L \cdot B$ is locally simple and therefore coincides with L . Therefore L is a splitting subfield in \overline{F} . Hence L is invariant 36.

Conversely, let L is embedded to \overline{F} as invariant subfield. Then from statement 36 it follows that it is a splitting subfield. Therefore L is normal over F by the definition. \square

It should be noted that most of all results about differential closure hold for normal extensions.

Statement 39. *Let L be a splitting field and let B be a simple differentially finitely generated subring in L . Then there exists a homomorphism mapping splitting field F of B to L . Among the subfields in L homeomorphic to F there exists a largest one which coincides with the subfield generated by all rings isomorphic to B .*

Proof. We need to map the field F to L . We can identify L with a splitting subfield in its differential closure. Then L is invariant. In other words, every algebra isomorphic to B is in L . Therefore the field generated by all algebras isomorphic to B is in L . From statement 34 it follows that our field is isomorphic to F . From the definition it coincides with the maximal subfield in L isomorphic to F . \square

If L is a splitting field and F is the largest subfield in L isomorphic to a splitting field, then F will be called a splitting subfield. From statements above easily follows the following corollary.

Corollary 40. *Every splitting field is normal over its splitting subfield.*

Statement 41. *Let L be arbitrary splitting field and F be its splitting subfield. Then F is invariant under action of the group of all differential automorphisms of L over K .*

Proof. The proof immediately follows from statement 39. \square

A differential Galois group $\text{Gal}^\Delta(L/F)$ of normal extension $F \subseteq L$ is a group of all differential automorphisms of L over F .

Statement 42. *Subfield F of a splitting field L is invariant under $\text{Gal}^\Delta(L/K)$ iff it is a splitting subfield of some family of ring.*

Proof. The proof immediately follows from statement 39. \square

Statement 43. *Let $F \subseteq L$ be a normal extension of fields. Then $F = L^{\text{Gal}^\Delta(L/F)}$.*

Proof. The proof follows from statement 32 with F instead of K . \square

7 Galois correspondence for normal extensions

It should be recalled that we are building the theory under assumption that all rings are the algebras over a field K . The natural choice of the field K is the field of rational numbers \mathbb{Q} .

Let $F \subseteq L$ be a normal extension of fields, in other words, L is a splitting field over F . Denote by \mathcal{F} the set of all “good” subfields if L containing F . Let denote the following family of groups

$$\mathcal{G} = \{ H \subseteq \text{Gal}^\Delta(L/F) \mid H = \text{Gal}^\Delta(L/L^H), L^H \in \mathcal{F} \}.$$

The set $\{L^H \mid H \subseteq \text{Gal}^\Delta(L/F)\}$ will be denoted by \mathcal{F}' . Let \mathcal{N} denotes the set of all invariant under $\text{Gal}^\Delta(L/F)$ subfields.

Note that the set \mathcal{F}' does not necessarily contain all intermediate subfields between F and L , and the set \mathcal{G} does not contain all subgroups in $\text{Gal}^\Delta(L/F)$ even in non differential case. Additionally, let us note that generally speaking the inclusion $\mathcal{F} \subseteq \mathcal{F}'$ is strict. Also, there is an inclusion $\mathcal{N} \subseteq \mathcal{F}$. We shall give all examples latter.

Theorem 44. *Under notations above the following conditions hold*

1. *The mappings $F \mapsto \text{Gal}^\Delta(L/F)$ and $H \mapsto L^H$ are inverse to each other bijections between the sets \mathcal{F} and \mathcal{G} .*
2. *The mentioned correspondence provide a bijection between \mathcal{N} and the set of all normal subgroups in \mathcal{G} , and for any invariant subfield F' the following equality holds*

$$\text{Gal}^\Delta(F'/F) = \text{Gal}^\Delta(L/F)/\text{Gal}^\Delta(L/F')$$

3. *The cardinality of the set of all extensions of an automorphism*

$$\varphi: F \rightarrow F$$

to automorphism of L does not depend on the choice of φ .

Proof. (1). The equality $H = \text{Gal}^\Delta(L^H/F)$ follows from the definition of \mathcal{G} and the equality $F = L^{\text{Gal}^\Delta(L/F)}$ follows from statement 43.

(2). If subgroup H is normal, then straightforward calculation shows that the field L^H is invariant. Conversely, every invariant subfield F' is “good” (statements 42 and 35) and therefore belongs to \mathcal{F} . Besides the group $H = \text{Gal}^\Delta(L/F')$ have the following property $F' = L^H$ (statement 43). Further is obvious.

(3). Let $\overline{\varphi}$ be an extension of φ to L , then it is easy to see that $\overline{\varphi}\text{Gal}^\Delta(L/F)$ is the set of all extensions of φ . \square

Example 45. The following example is taken from [7] and [3]. We shall show that behavior of a normal extension and its group of automorphism can be very sophisticated.

We amount to nothing more than the case of one derivation. Let K be an algebraically closed field of constants. Consider the equation $y' = y^3 - y^2$. Then from corollary [7, p. 532] it follows that its splitting field is

$$L = K(x_1, \dots, x_n, \dots),$$

where x_k are algebraically independent over K and $x'_k = x_k^3 - x_k^2$. It is easy to see that its differential Galois group is the group of all permutations of x_k , in other words, it coincides with the group of all permutations of natural numbers $S_{\mathbb{N}}$.

The subfields of the form $K(x_{i_1}, \dots, x_{i_n})$ (generated by finitely many number of elements x_i) are “good” and belongs to \mathcal{F} . The field

$$K(x_2, x_4, \dots, x_{2n}, \dots)$$

is in $\mathcal{F}' \setminus \mathcal{F}$. Indeed, it coincides with the field of invariant elements for the group permuting x_k with odd indexes. But this field is isomorphic

to L and thus is a splitting field. Therefore L is not constructed over our field. The subfield $K(x_2, x_3, \dots, x_n, \dots)$ does not belong even to \mathcal{F}' because every permutation leaving all elements x_k stable except possibly one is the identity permutation.

Let us denote a subgroup $A_{\mathbb{N}}^F$ in $S_{\mathbb{N}}$ as follows. The permutation belongs to $A_{\mathbb{N}}^F$ iff σ permute only finitely many elements by even permutation. It is easy to see that this group is the smallest normal subgroup in $S_{\mathbb{N}}$. But its field of invariant elements coincides with K . In other words, for any element $f \in L$ using addition, subtraction, multiplication, division, derivation we shall get arbitrary element of the field L .

8 Connection with differential algebraic varieties

Let B be an arbitrary differentially finitely generated algebra over K . Let \overline{K} be its differential closure. By $X_{\overline{K}}$ we shall denote the set of all \overline{K} points of algebra B . In other words, $X_{\overline{K}}$ is the set of all differential homomorphisms of B to \overline{K} over K . As before G will denote the group $\text{Gal}^{\Delta}(\overline{K}/K)$. The group G acts on $X_{\overline{K}}$ by the natural way $\xi \mapsto g \circ \xi$, $g \in G$, $\xi \in X_{\overline{K}}$. We shall show the relation between $\text{SMax}^{\Delta} B$ and $X_{\overline{K}}$.

Theorem 46. *Under assumptions above the following holds*

$$\text{SMax}^{\Delta} B = X_{\overline{K}}/G.$$

Proof. Let us construct a mapping from $X_{\overline{K}}$ to $\text{SMax}^{\Delta} B$, namely, each ξ goes to $\ker \xi$. From statement 26 it follows that the field \overline{K} is constructed over K and therefore statement 20 implies that $\xi(B)$ is locally simple. Therefore the mapping is well-defined. Since every differentially finitely generated locally simple algebra can be embedded to \overline{K} , the mapping is surjective. From the definition of G group action our mapping preserves the orbits of group. Hence we have the following mapping

$$X_{\overline{K}}/G \rightarrow \text{SMax}^{\Delta} B.$$

We only need to check that this mapping is injective. Indeed, Let

$$\xi_1, \xi_2: B \rightarrow \overline{K}$$

have the same kernels. Let us denote the images of B under ξ_1 and ξ_2 by B_1 and B_2 , respectively.

$$\begin{array}{ccccc} B_1 & \subseteq & \overline{K} & \supseteq & B_2 \\ & \searrow \xi_1 & & \nearrow \xi_2 & \end{array}$$

Then the algebra B_1 is isomorphic to B_2 , this isomorphism can be extended to the isomorphism of its fraction fields F_1 and F_2 . From statement 22 the field \overline{K} is constructed over F_1 and F_2 and coincides with their differential closure. Therefore from corollary 26 this isomorphism can be extended to an automorphism g of the field \overline{K} . From the definition of g it follows that $\xi_2 = g \circ \xi_1$. \square

References

- [1] M. F. Atiyah, I. G. Macdonald. Introduction to commutative algebra. Addison-Wesley. 1969.
- [2] E. R. Kolchin. Differential Algebra and Algebraic Groups. Academic Press, New York, 1976.
- [3] E. R. Kolchin. Constrained Extensions of Differential Fields. Advances in Math, 12, 1974, pp. 141-170.
- [4] M. van der Put and M. F. Singer. Galois Theory of Linear Differential Equations Grundlehren der mathematischen Wissenschaften, Volume 328, Springer, 2003.
- [5] P. J. Cassidy and M. F. Singer. Galois theory of parameterized differential equations and linear differential algebraic groups. Differential Equations and Quantum Groups (IRMA Lectures in Mathematics and Theoretical Physics Vol. 9), ed. D. Bertrand, B. Enriquez, C. Mitschi, C. Sabbah, R. Schaeffke, EMS Publishing house, pp. 113-157, 2006.
- [6] B. Poizat, M. Klein. A course in model theory: an introduction to contemporary mathematical logic. Springer, 2000
- [7] M. Rosenlicht. The nonminimality of the differential closure. Pacific J. Math., vol. 52, no. 2, 1974, pp. 529-537
- [8] J. J. Kovacic. The differential Galois theory of strongly normal extensions. Trans. AMS, Vol 355, Number 11, 2003, pp. 4475-4522.